Problem 4.40

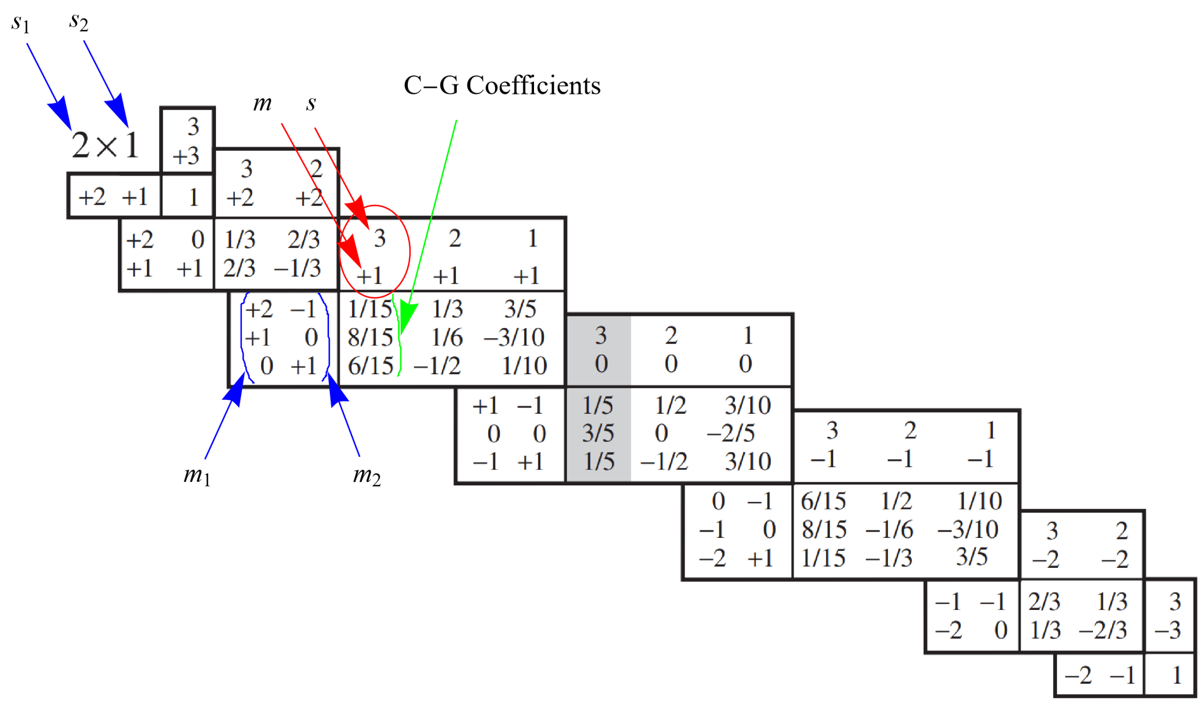
(a) A particle of spin 1 and a particle of spin 2 are at rest in a configuration such that the total spin is 3, and its z component is $\hbar$. If you measured the z-component of the angular momentum of the spin-2 particle, what values might you get, and what is the probability of each one? *Comment:* Using Clebsch–Gordan tables is like driving a stick-shift—scary and frustrating when you start out, but easy once you get the hang of it.

(b) An electron with spin down is in the state ψ_{510} of the hydrogen atom. If you could measure the total angular momentum squared of the electron alone (*not* including the proton spin), what values might you get, and what is the probability of each?

Solution

Part (a)

The total spin is 3, so $s = 3$. The z-component of spin angular momentum of the system is $1\hbar$, so $m = 1$. Since the particles have spin 1 and spin 2, use the 2×1 table of Clebsch–Gordon coefficients below to expand $|3\,1\rangle$ in terms of the eigenstates of the z-component of the angular momentum of the spin-2 particle.



$$|s\,m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1\, s_2\, m_1\, m_2\rangle$$

$$|3\,1\rangle = \sqrt{\frac{1}{15}}\,|2\,1\,2\,\,-1\rangle + \sqrt{\frac{8}{15}}\,|2\,1\,1\,0\rangle + \sqrt{\frac{6}{15}}\,|2\,1\,0\,1\rangle$$

Since $s_1 = 2$, apply $S_z^{(1)}$ to each of the eigenstates $|s_1\, s_2\, m_1\, m_2\rangle$ to determine the possible measurements of the z-component of the angular momentum of the particle with spin 2.

$$S_z^{(1)}|s_1\, s_2\, m_1\, m_2\rangle = m_1\hbar\,|s_1\, s_2\, m_1\, m_2\rangle \quad \rightarrow \quad \begin{cases} S_z^{(1)}|2\,1\,2\,\,{-1}\rangle = 2\hbar\,|2\,1\,2\,\,{-1}\rangle \\ S_z^{(1)}|2\,1\,1\,0\rangle = 1\hbar\,|2\,1\,1\,0\rangle \\ S_z^{(1)}|2\,1\,0\,1\rangle = 0\hbar\,|2\,1\,0\,1\rangle \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$2\hbar \quad \text{with probability} \quad \left|\sqrt{\frac{1}{15}}\right|^2 = \frac{1}{15}$$

$$\hbar \quad \text{with probability} \quad \left|\sqrt{\frac{8}{15}}\right|^2 = \frac{8}{15}$$

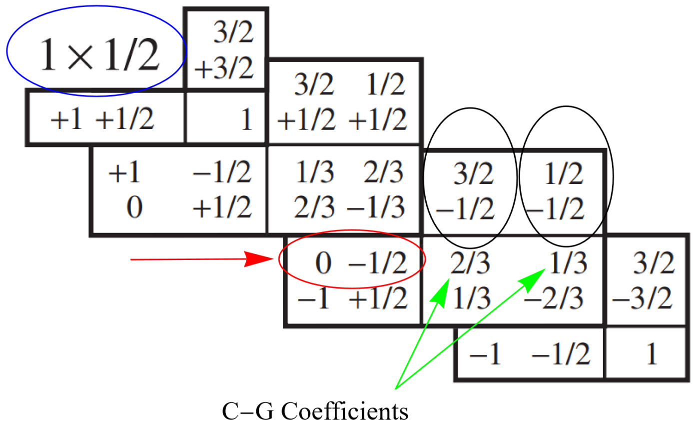
$$0 \quad \text{with probability} \quad \left|\sqrt{\frac{6}{15}}\right|^2 = \frac{6}{15}.$$

Part (b)

Construct the electron's wave function from the given information. Then expand it in terms of the eigenstates of the total angular momentum squared by using the appropriate Clebsch–Gordon table.

$$\begin{aligned}
\boldsymbol{\Psi}_e(r,\theta,\phi) &= \psi_{510}(r,\theta,\phi)\chi_- \\
&= R_{51}(r)Y_1^0(\theta,\phi)\chi_- \\
&= R_{51}(r)|1\,0\rangle\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle \\
&= R_{51}(r)\left|1\,\frac{1}{2}\,0\,\frac{-1}{2}\right\rangle \\
&= R_{51}(r)\left(\sqrt{\frac{2}{3}}\left|\frac{3}{2}\,\frac{-1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2}\,\frac{-1}{2}\right\rangle\right)
\end{aligned}$$

$$\left|1\ \frac{1}{2}\ 0\ \frac{-1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle$$



Now apply J^2 to each of the eigenstates $|j\,m\rangle$ to determine the possible measurements of the total angular momentum squared of the electron alone.

$$J^2|j\ m\rangle = \hbar^2 j(j+1)\,|j\ m\rangle \quad \rightarrow \quad \begin{cases} J^2\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle = \frac{15\hbar^2}{4}\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle \\ J^2\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle = \frac{3\hbar^2}{4}\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle \end{cases}$$

Therefore, the possible measurements and their corresponding probabilities are

$$\frac{15\hbar^2}{4} \quad \text{with probability} \quad \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$$

$$\frac{3\hbar^2}{4} \quad \text{with probability} \quad \left|\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{3}.$$